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MODELS FOR SITING REPAIR PARTS INVENTORIES IN SUPPORT OF A NAVAL AIR REWORK FACILITY

by

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April 1981

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ABSTRACT

Trade-off models are developed for siting inventories of repair parts by Naval Supply Center (NSC) which must support a local Naval Air Rework Facility (NARF). Two strategies are considered; siting at the NSC with direct delivery to the NARF and siting at the NARF. Three direct delivery alternatives which include both scheduled and unscheduled delivery schemes are modeled when siting is at the NSC. The measure of effectiveness for all alternatives is the expected total costs per time period. Cost elements include delivery costs and production delay costs. Algorithms for solving the trade-off models are also presented.

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I. INTRODUCTION

In the consolidation of wholesale supply support between the Naval Supply Centers (NSC) at Oakland, San Diego, and Norfolk, and the neighboring Naval Air Station (NAS) supply centers, the question of supply support of the local Naval Air Rework Facilities (NARFs) is of major concern. [1] A goal of these consolidations is not to degrade service to the NARF and hence the question of whether special inventories should be located at the customer's site is raised. One obvious advantage would be that the response time of the supply system would be quicker because the travel time would be shorter from the on-site system than it would be if the material were held at the NSC. The transportation costs and customer delay costs would also be less. The disadvantage is that additional costs are incurred in maintaining these on-site systems.

The answer to this question of supply support lies in the results of a trade-off analysis which seeks to find a balance between the three cost components mentioned above. Such a balance may result in all the inventory at the customer site and none at the NSC, or all at the NSC and none on-site, or some at each location. The purpose of this report is to develop the models needed to conduct the trade-off analysis for the all-or-none situations since they correspond to the current philosophy of the NSCs towards NARF support.

Figure 1 suggests the details which should be considered in the trade-off model. The element denoted as OSIS is the on-site inventory system.

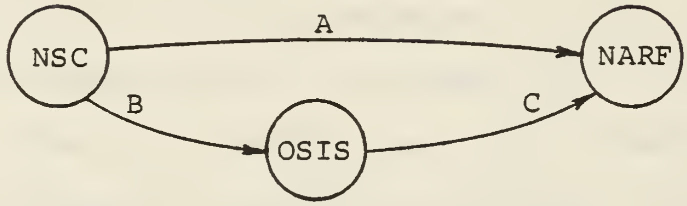

Figure 1.

Branch A of the figure represents the alternative of direct delivery from the NSC to the NARF. Branches B and C represent the alternative of a periodic replenishment to the OSIS from the NSC and direct delivery as each demand occurs from the OSIS to the NARF, respectively. Branch A deliveries may not be immediate upon receipt of a demand in contrast to Branch C. In fact, the most economical approach may be to delay delivery until several demands have been received.

Deterministic Demand--To develop a basis for understanding the more complex probabilistic direct delivery models to be presented later, we consider the case of deterministic or known demand. Suppose that a truck has a capacity of n units of an item. Suppose also that C_T is the cost of a trip by the truck from the NSC to the NARF. If the truck makes a trip as soon as the NSC receives and processes a unit demand, then the cost of shipping the unit is C_T. If, however, the truck waits for k units, then the shipping cost per unit is

$$\frac{C_T}{k}, \tag{1}$$

and obviously the cheapest unit cost occurs when the truck is filled. That is,

$$\frac{C_T}{n}.$$

However, while the truck is waiting for a full load, the units waiting will cause delays in production and these delays will result in extra costs to the NARF. To model these delays, let us suppose that a unit is needed every t units of time because of the repair schedule and that the cost of a delay over t for one unit is C_D. If we wait for k units to be accumulated, the total delay cost will be

$$\frac{C_D\, k\,[k-1]}{2} \tag{2}$$

To confirm formula (2), consider Figure 2.

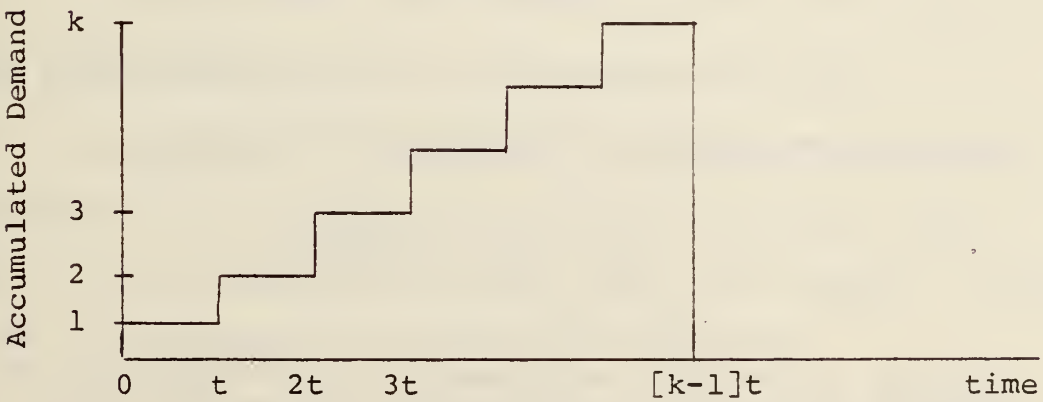

Figure 2.

If we wait until k units have accumulated, the truck will not leave until time $[k-1]t$. The first unit demanded occurs at time = 0 and is delayed $[k-1]$ periods of length t, the second unit demanded occurs at time t, and is delayed $[k-2]$ periods, the third unit is demanded at time $2t$, and waits $[k-3]$ periods, and so on. Only the k^{th} unit doesn't wait. The total waiting time in periods is therefore

$$[k-1] + [k-2] + [k-3] + --- + 1 + 0 \ . \tag{3}$$

The sum given by (3) can be written is a short form as

$$\frac{k[k-1]}{2} \ , \tag{4}$$

and when we multiply (4) by the cost C_D per unit delayed one period we get formula (2).

The average delay cost per unit is then obtained by dividing (2) by k; that is,

$$\frac{C_D[k-1]}{2} \ . \tag{5}$$

The total average cost of shipping and delay can now be written as the sum of formulas (1) and (5):

$$C(k) = \frac{C_T}{k} + \frac{C_D[k-1]}{2} \ . \tag{6}$$

Equation (6) can also be viewed as the average cost per period because demand is a known function of time.

Random Demand--Equation (6) was derived based on the assumption that one unit of a repair part is demanded every t time units. This demand pattern would correspond to a repair part which is replaced in every component undergoing overhaul. If, however, a repair part is needed only p percent of the time in such a component, where $p < 100\%$, then such a part may not be demanded every t unit of time and we must therefore consider minimizing the expected total costs per period. And since it may not be desirable for the truck to always be filled when it makes a trip, we need to consider some alternative delivery strategies. Those which seem most appropriate are:

1. The truck makes a delivery at the end of every N periods if there is at least one demand during the N periods.
2. The truck makes a delivery as soon as demands have accumulated to a specified number K not exceeding its capacity n.
3. The truck makes a delivery in the $(N-1)^{st}$ period following the first demand received after the last delivery.

Alternative 1 corresponds to scheduled deliveries. Alternative 3 is a variant of scheduled deliveries where time = 0 corresponds to the time of the first demand. Alternative 2 corresponds to unscheduled deliveries.

To compare these alternatives, we need to develop expressions for the expected average total costs per period

because the time between deliveries may be different for each alternative and thus the number delivered over a given time interval can be expected to be different. Chapters II, III, and IV present the derivations needed.

Another set of models are needed to complete the picture depicted in Figure 1. These are the models which form the basis for comparing the expected total costs per period of operating an on-site inventory system with those costs derived in Chapters II, III and IV. The derivations are presented in Chapter V.

Chapter VI combines the models from Chapters II through V into a structure such that the trade-off analyses can be made. Included in that structure is the impact of constraints imposed by time standards and truck capacity. These constraints are derived in Chapters II, III and IV for the direct delivery alternatives. The final chapter, Chapter VII, summarizes the modeling efforts and the results of the analysis. It also makes suggestions for model refinements.

II. ALTERNATIVE 1

Introduction--Alternative 1 assumes that a scheduled delivery will be made of all the units of an item needed to meet the demands which have accumulated by the end of N periods. If none have accumulated the scheduled delivery will be cancelled.

Probabilities--We assume that only one unit of a given repair part is needed by a component undergoing repair. We also assume that the probability that it will need to be replaced is a constant denoted by p. Therefore, the probability distribution of the demand for x units of the repair part during an interval of N periods (during which N components are repaired) is described by the binomial distribution; that is,

$$p(x;N) = \binom{N}{x} p^{x} (1-p)^{N-x} , \tag{7}$$

where $x = 0, 1, 2, \ldots, N$.

However, for this delivery alternative we would make a delivery only if at least one demand occurred during the N periods. Therefore, we must condition (7) for at least one demand before we can talk about the expected shipping and shortage costs. Equation (8) provides the needed form.

$$p(x;N, x\geq 1) = \binom{N}{x} \frac{p^{x} (1-p)^{N-x}}{[1 - (1-p)^{N}]} \tag{8}$$

where $x = 1, 2, 3, \ldots, N$.

Cost Elements--As with the deterministic model presented in Chapter I, the costs which will be considered in this and the other delivery alternatives are the costs associated with delivering the item to the NARF and the costs of delay associated with not delivering the item as soon as it is demanded.

Let C_T represent the round-trip costs of a delivery. These will include the costs of the truck and driver from the time the truck starts being loaded at the NSC until it returns to the NSC. C_T will be incurred each time a delivery takes place. If, under the scheduled delivery scheme, a demand does not occur during an interval of N periods then we assume initially that no penalty cost π is incurred for cancelling a delivery. Later, we will incorporate that cost into the model.

The delay costs are costs incurred at the NARF as a consequence of not having a needed part on hand at the instant it is needed. Two elements need to be considered here. The first is a cost, denoted by S, which is the cost associated with putting the component aside. This could include putting it on a shelf in a storage area and documenting how far repair had progressed and what had been ordered. The second element, denoted by C_D, is a time dependent cost which we will refer to as the delay cost per period. This element might include labor costs due to work stoppage, inventory holding costs, and costs associated with a repaired component not being available to a customer such as a fleet squadron. In the initial

formulation of the expected delay costs we will include only the C_D element.

To develop the expected total costs per period, we must first develop the expected total costs for a time interval of N periods, given that at least one demand occurs and hence that a delivery will be made. We have already identified C_T to be the round-trip delivery cost. We must now determine the expression for the expected total delay costs. This will be the product of C_D and the expected total delays.

Expected total delays--The delays are a function of the number of different configurations that demands can take over N periods. For example, if $N = 2$ then there are three possible configurations. The one where a demand occurs in the first period and none occurs in the second results in a delay of one period. For the configuration having a demand in the second period and none in the first, a delay of zero periods results. The final configuration when $N = 2$ is a demand in both periods. This creates a delay of one period for the first period's demand and zero delay for the second period's demand, resulting in a total delay of one period.

The total number of configurations where exactly x demands occur is

$$n_x = \binom{N}{x} .$$

As a consequence, the total number of configurations which can occur over N periods and which have at least one demand is

$$n = \sum_{x=1}^{N} n_x \sum_{x=1}^{N} \binom{N}{x} = 2^N - 1 \ . \tag{9}$$

Theorem II-1. The expected total delay time for Alternative 1 is:

$$ETD(N) = \frac{N(N-1)p}{2[1 - (1-p)^N]} , \tag{10}$$

Proof. To determine the expected total delay associated with the n configurations given by (9) we first consider only those having exactly x demands where $x \geq 1$. The probability of each such configuration is:

$$P(x,N) = \frac{p^x \ (1-p)^{N-x}}{1 - (1-p)^N} \tag{11}$$

The number of configurations having a demand in period $1 \leq j \leq N$ is

$$m = \binom{N-1}{x-1} \ . \tag{12}$$

It is important to note that m is independent of j. Those demands occurring in period j will have to wait until period N for delivery and hence each must wait $N-j$ periods. The total of all delays for those configurations having x demands is therefore:

$$TD(x,N) = \sum_{j=1}^{N} \binom{N-1}{x-1} [N-j] =$$

$$= \binom{N-1}{x-1} \sum_{j=1}^{N} (N-j)$$

$$= \frac{N(N-1)}{2} \binom{N-1}{x-1} \tag{13}$$

From (11) and (13) it easily follows that the total expected delays over all x values is given by (14) which is identical to (10) and the proof is complete.

$$ETD(N) = \sum_{x=1}^{N} TD(x,N)P(x,N)$$

$$= \frac{N(N-1)p}{2[1 - (1-p)^N]} \sum_{x=1}^{N} \binom{N-1}{x-1} p^{x-1} (1-p)^{N-x}$$

$$= \frac{N(N-1)p}{2[1 - (1-p)^N]} \tag{14}$$

<u>Expected Total Costs per Period</u>--In developing the expected costs per period, we must consider a cycle which can include not making a delivery since we stated earlier that Alternative 1 has a scheduled delivery at the end of N periods if there is at least one demand during this time interval.

<u>*Theorem* II-2</u>. The expected total costs per period for Alternative 1 are:

$$\mathrm{ECP}(N) = \left[\frac{C_T[1 - (1-p)^N]}{N} + \frac{C_D(N-1)p}{2}\right]\left[\frac{-\ln[1 - (1-p)^N]}{(1-p)^N}\right] \tag{15}$$

Proof: We begin with time zero and assume that at least one demand occurs in the first N periods. The total expected costs over these N periods is then made up of the delivery cost, C_T, for one trip and the total expected delay costs which is the product of C_D and equation (14). The cost per period is obtained by dividing each term by N. The result is given as formula (16).

$$\frac{C_T}{N} + \frac{C_D(N-1)p}{2[1 - (1-p)^N]} \tag{16}$$

The associated probability of at least one demand during the first N periods is $1 - (1-p)^N$.

Next we assume no demands occur during the first N periods and at least one occurs during the next N periods. The expected cost per period is then:

$$\frac{C_T}{2N} + \frac{C_D(N-1)p}{4[1 - (1-p)^N]} .$$

The associated probability of no demands in the first N periods and at least one in the second N periods is

$$(1-p)^N [1 - (1-p)^N] .$$

The general formulas for no demands during $(k-1)N$ periods and at least one during the last N periods are (17) and (18).

$$\frac{C_T}{kN} + \frac{C_D(N-1)p}{2k[1 - (1-p)^N]} \; . \tag{17}$$

$$(1-p)^{(k-1)N} \, [1 - (1-p)^N] \; . \tag{18}$$

The expected costs per period over all k values is therefore

$$ECP(N) = \sum_{k=1}^{\infty} \left[\frac{C_T}{kN} + \frac{C_D(N-1)p}{2k[1 - (1-p)^N]} \right] (1-p)^{(k-1)N} [1 - (1-p)^N]$$

$$= \left[\frac{C_T[1-(1-p)^N]}{N} + \frac{C_D(N-1)p}{2} \right] \sum_{k=1}^{\infty} \frac{1}{k} (1-p)^{(k-1)N} \; . \tag{19}$$

The summation term of (19) can be rewritten as follows:

$$\sum_{k=1}^{\infty} \frac{1}{k} (1-p)^{(k-1)N} = \frac{1}{(1-p)^N} \sum_{k=1}^{\infty} \frac{1}{k} (1-p)^{kN}$$

$$= \frac{1}{a} \sum_{k=1}^{\infty} \frac{a^k}{k}$$

$$= \frac{1}{a} [-\ell n(1-a)] \; , \tag{20}$$

where $a = (1-p)^N$. The series sums to the negative of $\ell n(1-a)$ since $a < 1$.

When (20) is combined with (19) the result is (15) and the proof is complete.

Determination of Optimal N--Because N takes on only integer values, the use of finite differences is appropriate for determining optimal N. Optimal N is that value of N which satisfies the following relation:

$$ECP(N-1) > ECP(N) \leq ECP(N+1) .$$

Equivalently, optimal N is the largest value of N such that

$$\Delta ECP(N) = ECP(N) - ECP(N-1) < 0 .$$

Unfortunately, the algebraic expression for $\Delta ECP(N)$ which results from using (15) and taking the difference between $ECP(N)$ and $ECP(N-1)$ is just as complex as (15). Therefore, a numerical evaluation of (15) for a range of N values appears to be the most practical way of searching for optimal N.

Penalty from Cancelling a Scheduled Delivery--Suppose that the cost of delivery includes a penalty cost π for cancelling a scheduled delivery (because no demands occurred during an interval of N periods). If no demands occur during the first $(k-1)N$ periods then $k-1$ cancellations will occur. Since the cost for each is π, equation (17) will be modified to that given by (21).

$$\frac{(k-1)\pi}{kN} + \frac{C_T}{kN} + \frac{C_D(N-1)p}{2k[1-(1-p)^N]} \tag{21}$$

The expected costs per period over all k values is then

$$ECP(N) = \frac{\pi}{N} + \left[\frac{(C_T-\pi)[1-(1-p)^N]}{N} + \frac{C_D(N-1)p}{2}\right]\left[\frac{-\ell n[1-(1-p)^N]}{(1-p)^N}\right] . \tag{22}$$

The impact of introducing π can be easily seen by comparing (22) with (15).

<u>Delay Cost Independent of Time</u>--Suppose that there is a fixed unit cost S associated with putting aside a component until the needed part arrives. The total of such costs during an interval of N periods in which at least one demand occurs is the product of π and the expected number of units being delivered. The expected number of units being delivered, M_1, is given by (23).

$$\begin{aligned} M_1 &= \sum_{x=1}^{N} x \frac{\binom{N}{x} p^x (1-p)^{N-x}}{[1-(1-p)^N]} \\ &= \frac{E(x)}{[1-(1-p)^N]} \\ &= \frac{Np}{[1-(1-p)^N]} \end{aligned} \tag{23}$$

Here E(x) is the expected value of a binomial random variable.

The total expected cost over N is then increased by the product SM_1. Formula (17) is then modified as follows

$$\frac{C_T}{kN} + \frac{Sp + C_D(N-1)p}{2k[1 - (1-p)^N]} \quad , \tag{24}$$

and (15) is therefore modified as shown by (25).

$$ECP(N) = \left[\frac{C_T[1-(1-p)^N]}{N} + \frac{C_D(N-1)p}{2} + \frac{Sp}{2}\right]\left[\frac{-\ln[1-(1-p)^N]}{(1-p)^N}\right] \tag{25}$$

If the time-dependent delay cost C_D is negligible relative to S then optimal N will be infinite since the S term in the left bracket of (24) is not an increasing function of N.

<u>Time Standards</u>--Time standards have been established as upper bounds on the average time allowed a supply center to respond to a customer's demand. If T denotes this standard in periods then the expected delay in periods per unit delivery is required to not exceed T.

<u>*Theorem* II-3</u>. The expected delay per unit under alternative 1 is

$$UD(N) = \frac{N-1}{2} \quad . \tag{26}$$

<u>*Proof*</u>. Equation (13) gives the total of all delays for those configurations having x demands during N periods. Equation (11) gives the probability of each such configuration. Dividing (13) by x gives the total of all average delays per unit for those configurations. Thus the expression for $UD(N)$ is

$$UD(N) = \sum_{x=1}^{N} \frac{TD(x,N)}{x} P(x,N)$$

$$= \frac{N(N-1)p}{2[1 - (1-p)^N]} \sum_{x=1}^{N} \frac{1}{x} \binom{N-1}{x-1} p^{x-1} (1-p)^{N-x}$$

$$= \frac{(N-1)}{2[1 - (1-p)^N]} \sum_{x=1}^{N} \binom{N}{x} p^x (1-p)^{N-x}$$

$$= \frac{(N-1)}{2[1 - (1-p)^N]} [1 - (1-p)^N]$$

Cancellation of the probability terms in the denominator and numerator results in the right side of (26) and the proof is complete.

Setting up the time constraint inequality; namely,

$$UD(N) = \frac{N-1}{2} \leq T \quad ;$$

and solving for N gives

$$N \leq 2T + 1 \tag{27}$$

Inequality (27) provides an easy way of determining if an optimal N is also feasible with respect to a given time standard.

<u>Truck Capacity Constraint</u>--When C_T was introduced it was assumed to represent the round-trip delivery costs incurred by one truck (and driver). As a consequence, it is appropriate to include a constraint to represent the capacity of that

truck. Let M represent the maximum number of units of a given repair part that can be loaded on a given truck. Under alternative 1 it is possible to have a demand during each of a sequence of N periods. As a consequence, N units would need to be delivered. Therefore, $N \leq M$ is required to insure that all possible quantities demanded during N periods can be delivered by the truck at the end of these periods.

III. ALTERNATIVE 2

Introduction--Alternative 2 assumes that a delivery is not made until K units of an item have been demanded. Delivery is assumed to take place as soon as the last demand occurs. The basic assumptions spelled out in Chapter II with respect to cost elements and the probability of demand for a repair part also apply to this alternative.

Probabilities--For Alternative 2 we will be interested in the probability that n periods will pass before we accumulate K demands with the last demand occurring in period n. This probability is described by the negative binomial distribution [2]; namely,

$$p(n;\ K) = \binom{n-1}{K-1} p^K (1-p)^{n-K} \tag{28}$$

where $n=K,\ K+1,\ K+2,\ \ldots$.

Expected Total Delays--As was observed in Chapter II, the delays are a function of the number of different configurations that demands can take over n periods. When $K = 2$ and $n = 4$, we get 3 demand configurations. In all three, the second demand occurs in the fourth period. Thus, the first demand can occur in either period 1, 2, or 3. The delay is zero for the second demand in all three cases. The delay for the first demand is 3, 2, and 1 periods, respectively.

For the general case, the number of configurations, m, having a demand occurring in periods $1 \leq j \leq n - 1$ is given by (29).

$$m = \binom{n-2}{k-2} \tag{29}$$

Equation (29) differs from (12) because the K^{th} demand always occurs in period n for Alternative 2. Again we see that m is independent of j.

Theorem III-1. The expected total delay time for Alternative 2 is

$$ETD(K) = \frac{K(K-1)}{2p} \quad , \tag{30}$$

Proof. The total delay associated with the configurations having a demand in period j is $m[n-j]$. The total of all delays for K demands during n periods is therefore

$$TD(n, K) = \sum_{j=1}^{n-1} (n-j) \binom{n-2}{K-2} \ . \tag{31}$$

We see that (31) can be rewritten as

$$\begin{aligned} TD(n, K) &= \binom{n-2}{K-2} \sum_{j=1}^{n-1} (n-j) \\ &= \frac{n(n-1)}{2} \binom{n-2}{K-2} \\ &= \frac{n(K-1)}{2} \binom{n-1}{K-1} \end{aligned} \tag{32}$$

The probability of each configuration is

$$P(n,K) = p^K(1-p)^{n-K} \tag{33}$$

The total expected delays over all n values are obtained by summing the products of (32) and (33) over all $n \geq K$.

$$ETD(K) = \sum_{n=K}^{\infty} TD(n,K)P(n,K)$$

$$= \sum_{n=K}^{\infty} \frac{n(K-1)}{2} \binom{n-1}{K-1} p^K (1-p)^{n-K}$$

$$= \frac{K-1}{2} \sum_{n=K}^{\infty} n \binom{n-1}{K-1} p^K (1-p)^{n-K}$$

$$= \frac{K-1}{2} \sum_{x=0}^{\infty} [x + K] \binom{K+x-1}{K-1} p^K (1-p)^x$$

$$= \frac{K-1}{2} \left[K + \sum_{x=0}^{\infty} x \binom{K+x-1}{K-1} p^K (1-p)^x \right]$$

$$= \frac{K-1}{2} \left[K + \frac{K(1-p)}{p} \right] = \frac{K(K-1)}{2p} \tag{34}$$

Since (34) is identical to (30), the proof is complete.

Expected Total Costs per Period--Under Alternative 2 a delivery will be made as soon as K demands have accumulated. This will span $n \geq K$ periods.

Theorem III-2. The expected total costs per period for Alternative 2 are:

$$ECP(K) = C_T \sum_{n=K}^{\infty} \frac{1}{n} \binom{n-1}{K-1} p^K (1-p)^{n-K} + \frac{C_D(K-1)}{2} \quad . \tag{35}$$

Proof. For a specified value of K, the delivery costs per period are

$$\mathrm{ETCP}(K) = C_T \sum_{n=K}^{\infty} \frac{1}{n}\binom{n-1}{K-1} p^K (1-p)^{n-K} \; . \tag{36}$$

Equation (36) results from summing the product of C_T/n and (28) over all $n \geq K$.

The expected delays costs per period are obtained by first dividing the total delays for a given n (namely TD(n,K)) by n and then summing the product of C_D, that result, and (33) over all $n \geq K$.

$$\mathrm{EDCP}(K) = C_D \sum_{n=K}^{\infty} \frac{TD(n,K)}{n} P(n,K)$$

$$= C_D \frac{(K-1)}{2} \sum_{n=K}^{\infty} \binom{n-1}{K-1} p^K (1-p)^{n-K}$$

$$= C_D \frac{(K-1)}{2} \; . \tag{37}$$

Adding (36) and (37) together gives (35).

Determination of Optimal K--The use of finite differences to determine optimal K does not result in any relationship which is less complex than (35). In particular, the infinite sum in the C_T term remains. Fortunately, however, bounds can be derived for (35) which can serve to reduce the number of K values for which (35) must be evaluated in searching for optimal K.

Theorem III-3. The expected delivery cost per period for Alternative 2 is bounded as follows:

$$\frac{C_T p}{K} < \mathrm{ETCP}(K) < \frac{C_T p}{K-1} \quad . \tag{38}$$

Proof. Equation (36) is, in reality, the product of C_T and the expected value of 1/n; that is,

$$C_T E\left(\frac{1}{n}\right) \quad .$$

Jensen's Rule [2] therefore allows us to obtain the lower bound. That rules states that

$$\frac{1}{E(n)} \leq E\left(\frac{1}{n}\right) \quad .$$

Because C_T is a constant we can write

$$\frac{C_T}{E(n)} \leq C_T E\left(\frac{1}{n}\right) = \mathrm{ETCP}(K) \quad .$$

The value of $E(n)$, when n has the negative binomial distribution given by (28), is, from [2],

$$E(n) = \frac{K}{p} \quad . \tag{39}$$

Therefore,

$$\frac{C_T}{E(n)} = \frac{C_T p}{K} \quad . \tag{40}$$

Equation (40) is the lower bound of (38).

Next,

$$\sum_{n=K}^{\infty} \frac{1}{n}\binom{n-1}{K-1} p^K(1-p)^{n-K} = \sum_{n=K}^{\infty} \frac{n-1}{n} \binom{n-2}{K-1} p^K(1-p)^{n-K}$$

$$< \sum_{n=K}^{\infty} \binom{n-2}{K-1} p^K(1-p)^{n-k} ,$$

since

$$\frac{n-1}{n} < \frac{n}{n} = 1 .$$

Then,

$$\sum_{n=K}^{\infty} \binom{n-2}{K-1} p^K(1-p)^{n-K} = \frac{p}{K-1} \sum_{n=K}^{\infty} \binom{n-2}{K-2} p^{K-1}(1-p)^{n-K} = \frac{p}{K-1} .$$

Therefore,

$$E\left(\frac{1}{n}\right) < \frac{p}{K-1} ,$$

and hence the upper bound of (38) is (41)

$$\frac{C_T p}{K-1} . \tag{41}$$

When (38) is combined with (37), the bounds for the expected total costs per period are easily determined to be:

$$\frac{C_T p}{K} + \frac{C_D[K-1]}{2} \leq ECP(K) \leq \frac{C_T p}{K-1} + \frac{C_D[K-1]}{2} . \tag{42}$$

These bounds can be analyzed using finite differences to determine their respective optimal K values. Theorem III-4 and its proof give the results.

Theorem III-4. The value of K which minimizes the lower bound of (42) is one less in value than the value of K which minimizes the upper bound.

Proof. Using the approach of finite differences we know that optimal K for the lower bound is the largest K value for which

$$C(K) - C(K-1) < 0 .$$

When we evaluate this difference we get

$$C(K) - C(K-1) = -\frac{C_T p}{K(K-1)} + \frac{C_D}{2} < 0 ,$$

which can be rewritten as

$$K(K-1) < \frac{2C_T p}{C_D} . \tag{43}$$

Similarly, for the upper bound

$$C(K) - C(K-1) = -\frac{C_T p}{(K-1)(K-2)} + \frac{C_D}{2} < 0 ,$$

which can be rewritten as

$$(K-1)(K-2) < \frac{2C_T p}{C_D} . \tag{44}$$

We see immediately that (43) and (44) differ by the fact that K has been replaced by $K-1$. As a consequence, optimal K for (43) will be one unit less than optimal K for (44).

Conjecture. The value of K which minimizes ECP(K) is either K^* or K^{**} where K^* minimizes the lower bound of (42) and K^{**} minimizes the upper bound.

Although computational experience shows this conjecture to be true and intuition strongly favors it, no formal proof has been yet been discovered.

Modification of the Expected Costs per Period--In Chapter II, two modifications of the basic expected costs formula were presented. The first was the inclusion of a penalty cost for cancelling a scheduled delivery. Such a cost would not be appropriate to Alternative 2 since it is an unscheduled delivery strategy. However, it could be argued that C_T for this alternative might be appropriately higher than that for a scheduled delivery since the need for a truck will not be known until the Kth demand has occurred.

The second modification was the inclusion of a fixed delay cost S per unit demanded. Under Alternative 2, the total of such costs is merely SK since K units are always delivered. The expected costs per period is then modified to include the term

$$SK \sum_{n=K}^{\infty} \frac{1}{n} \binom{n-1}{K-1} p^K (1-p)^{n-K} .$$

Thus (35) is now

$$ECP(K) = [C_T + SK] \sum_{n=K}^{\infty} \frac{1}{n}\binom{n-1}{K-1} p^K (1-p)^{n-K} + \frac{C_D(K-1)}{2} . \tag{45}$$

The bounds for ECP(K) are modified also. The lower is now

$$\frac{C_T p}{K} + Sp + \frac{C_D(K-1)}{2} ,$$

and the upper is

$$\frac{C_T p}{K-1} + \frac{SpK}{K-1} + \frac{C_D(K-1)}{2} .$$

If the value of C_D is negligible with respect to S then the value of K which minimizes the lower bound is infinity. The value of K which minimizes the upper bound is also infinity. Therefore, optimal K minimizing ECP(K) must also be infinite.

<u>Time Standards</u>--Again denote T as the upper bound on the average delay time per unit.

Theorem III-5. The expected delay per unit under Alternative 2 is

$$UD(K) = \frac{(K-1)}{2p} . \tag{46}$$

Proof. Because K units will always be delivered we can get the expected delay per unit by merely dividing the total expected delay, given by (30), by K. The result is (46).

When the time constraint is introduced, we get

$$UD(K) = \frac{K-1}{2p} \leq T \ .$$

Solving for K in this inequality gives

$$K \leq 2pT + 1 \ . \tag{47}$$

<u>Truck Capacity Constraint</u>--If M is the maximum capacity of the truck then it follows that $K \leq M$ is the capacity constraint.

<u>Expected Number of Periods Between Deliveries</u>--When comparing Alternative 2 with the other alternatives it will be useful to kno the expected number of periods between deliveries. The formula was given earlier as (39); namely,

$$E(n) = \frac{K}{p} \ .$$

IV. ALTERNATIVE 3

Introduction--Under Alternative 3, we start counting time from when the first demand occurs after the delivery truck has returned and is again available for further deliveries. We then wait N-1 periods after the first demand before we deliver again. The basic assumptions spelled out in Chapter II relative to cost elements and the probability of demand for a repair part also apply to this alternative.

Probabilities--The probability of x demands in N periods, given that the first one always occurs in the first period, is

$$p(x;N) = \binom{N-1}{x-1} p^{x-1}(1-p)^{N-x} , \tag{48}$$

where $x = 1, 2, 3, \ldots, N$.

Expected Total Delays--We start counting delay time from the period when the first demand occurs. If we deliver N-1 periods later then the total delay for the first demand is N-1 periods. The delays associated with subsequent demands occurring in periods 2, 3, up to N, are a function of the different configurations of demand which can occur. For example, if N=2 then two cases occur; the first has a demand in period 1 and none in period 2, the second has a demand in both periods 1 and 2. In each case the first demand is delayed one period. In the first case, no subsequent delay occurs since no subsequent demand occurs. In the second case, the second demand incurs a delay of essentially zero since the truck leaves shortly after the demand occurs.

The total number of configurations having exactly x demands during N is

$$n_x = \binom{N-1}{x-1} . \tag{49}$$

The total number of configurations for a given N is then

$$n = \sum_{x=1}^{N} \binom{N-1}{x-1} = 2^{N-1} . \tag{50}$$

Theorem IV-1. The expected total delay time for Alternative 3 is:

$$ETD(N) = [N-1] \left[\frac{(N-2)}{2} p + 1 \right] \tag{51}$$

Proof. The number of configurations having a demand in period $2 \leq j \leq N$ and a total of x demands over the N periods is

$$m = \binom{N-2}{x-2} , \tag{52}$$

where $x = 2, 3, \ldots, N$. Obviously, m is independent of j. The delay for a demand occurring in period j is $N-j$.

The delay for demands occurring in the first period is $N-1$ and equation (49) gives the total number of configurations having x demands. The total of all delays for these x demands is

$$TD(x,N) = \left[\binom{N-1}{x-1}(N-1) + \sum_{j=2}^{N}\binom{N-2}{x-2}(N-j)\right]$$

$$= \left[\binom{N-1}{x-1}(N-1) + \binom{N-2}{x-2}\frac{(N-1)(N-2)}{2}\right]$$

$$= \left[\binom{N-1}{x-1}(N-1) + \binom{N-1}{x-1}\frac{(x-1)(N-2)}{2}\right]$$

$$= \left[\frac{N}{2} + \frac{(N-2)x}{2}\right]\binom{N-1}{x-1} . \tag{53}$$

The probability of one configuration having x demands in N periods, given that one occurs in the first period is

$$P(x,N) = p^{x-1}(1-p)^{N-x} . \tag{54}$$

The expected total delay is then obtained by summing the products of (53) and (54) over all $1 \le x \le N$.

$$ETD(N) = \sum_{x=1}^{N} TD(x,N)\ P(x,N)$$

$$= \sum_{x=1}^{N}\left[\frac{N}{2} + \frac{(N-2)x}{2}\right]\binom{N-1}{x-1} p^{x-1}(1-p)^{N-x}$$

$$= \frac{N}{2}\sum_{x=1}^{N}\binom{N-1}{x-1} p^{x-1}(1-p)^{N-x}$$

$$+ \frac{N-2}{2}\sum_{x=1}^{N} x\binom{N-1}{x-1} p^{x-1}(1-p)^{N-x}$$

$$= \frac{N}{2} + \frac{N-2}{2} \sum_{u=0}^{N-1} (u+1) \binom{N-1}{u} p^u (1-p)^{N-1-u}$$

$$= \frac{N}{2} + \frac{N-2}{2} \left[(N-1)\ p + 1 \right]$$

$$= (N-1) \left[1 + \frac{N-2}{2}\ p \right] .$$

The proof is complete.

Expected Total Costs per Period--In analyzing the expected costs per period we consider a cycle whose length is N plus the time span in periods between the time the truck becomes available for a delivery and the first demand occurs.

Theorem IV-2. The expected total costs per period for Alternative 3 are

$$ECP(N) = ETC(N) \left[\sum_{k=1}^{\infty} \frac{1}{(k-1)+N}\ p(1-p)^{k-1} \right] , \qquad (55)$$

where:

$$ETC(N) = C_T + C_D(N-1) \left[\frac{(N-2)}{2}\ p + 1 \right] . \qquad (56)$$

Proof. Equation (56) describes the expected total costs over the N periods. Now, if the first demand occurs in the first period after the truck becomes available then the total cost per period is ETC(N) divided by N since we deliver in the Nth period. The probability of a demand in the first period is p. If, instead, the first demand occurs in period 2, then the average expected cost per period is ETC(N) divided by N+1 and the probability of the first demand occurring in

period 2 is $(1-p)p$. The general forms for the average cost per period and the associated probability when the first demand occurs in period k are:

$$\frac{ETC(N)}{(k-1)+N} \quad \text{and} \quad p(1-p)^{k-1} .$$

The expected total costs per period are the sum of the products of these two general forms over all possible k values. The result is equation (55). The summation term of equation (55) cannot be written in a closed form, however, bounds can be stated.

Theorem IV-3. The expected total costs per period under Alternative 3 are bounded as follows:

$$ETC(N)\left[\frac{p}{1+(N-1)p}\right] \le ECP(N) \le ETC(N)\left[\frac{p}{1-p}\min\left\{\frac{1}{N}, (-\ell n p)\right\}\right] \tag{56}$$

Proof. We can write ECP(N) as the product of ETC(N) and the expected value of the reciprocal of $(N+k-1)$ where k has a geometric distribution. Now from Jensen's Rule [2] we know that

$$\frac{1}{E(N+k-1)} \le E\left(\frac{1}{N+k-1}\right) . \tag{57}$$

Next,

$$\begin{aligned} E(N+k-1) &= N-1 + E(k) \\ &= N-1 + \frac{1}{p} \\ &= \frac{1 + (N-1)p}{p} \end{aligned} \tag{58}$$

The product of the lower bound from Jensen's Rule and ETC(N) is the lower bound given by (56); namely,

$$\mathrm{ETC}(N)\ \frac{p}{1+(N-1)p}\ . \tag{59}$$

Consider next the actual expansion of the bracket term of (55).

$$E\ \frac{1}{N+k-1} = \frac{1}{N}\ p + \frac{1}{N+1}\ p\ (1-p) + \frac{1}{N+2}\ p\ (1-p)^2 + --- \ . \tag{60}$$

When N=1 the right side of (60) reduces to

$$p + \frac{p(1-p)}{2} + \frac{p(1-p)^2}{3} + \frac{p(1-p)^3}{4} + ---$$

$$= \frac{p}{a}\left[a + \frac{a^2}{2} + \frac{a^3}{3} + \frac{a^4}{4}\right]$$

$$= \frac{p}{a}\ [-\ell n(1-a)]$$

$$= \frac{p}{1-p}\ [-\ell np]\ . \tag{61}$$

Here we introduced a in place of (1-p) to simplify the middle two steps. When N=2 we get

$$\frac{1}{2}\ p + \frac{1}{3}\ p(1-p) + \frac{1}{4}\ p(1-p)^2 + ---$$

$$= \frac{p}{a^2}\left[\frac{a^2}{2} + \frac{a^3}{3} + \frac{a^4}{4} + ---\right]$$

$$= \frac{p}{a^2}\ [-\ \ell n(1-a) - a]\ .$$

For general N:

$$E\left(\frac{1}{N+k-1}\right) = \frac{p}{a^N}\left[-\ln(1-a) - \sum_{k=1}^{N-1}\frac{a^k}{k}\right] . \tag{62}$$

Equation (62) can be used in place of the infinite sum for computations of ECP(N).

An upper bound is suggested by (62); namely,

$$\frac{p}{(1-p)^N}[-\ln p] .$$

However, (61) provides a tighter bound since $1 - p > (1-p)^N$ for $N \geq 2$.

Inspection of (60) suggests another upper bound. If we keep all denominators at a value of N we get

$$\frac{p}{N}\left[1 + (1-p) + (1-p)^2 + (1-p)^3 + ---\right]$$

$$= \frac{p}{N}\left[\frac{1}{1-p}\right]$$

$$= \frac{p}{1-p}\left[\frac{1}{N}\right] . \tag{63}$$

The advantage of (63) over (61) is that it is a decreasing function of N whereas $-\ln p$ remains constant. When $p < 0.368$, (63) gives the best result. However, when p increases to 0.7 or 0.9, we need N values of at least 3 and 10, respectively before (63) is preferred. Thus, we can state, in general, that our bound should be

$$\frac{p}{1-p}\min\left\{\frac{1}{N}, -\ln p\right\} \tag{64}$$

And, when we form the product of (64) with ETC(N) we get

$$\mathrm{ETC}(N)\left[\frac{p}{1-p}\min\left\{\frac{1}{N}, (-\ell n p)\right\}\right]$$

which is the upper bound of (56).

While (62) is not difficult to use when N is small, the summation becomes tedious for large N and hence optimal solutions to the bounds should be valuable in narrowing the values of N to be investigated. As in Chapter III, we have not been able to prove it but computational experience suggests that

Conjecture. The value of N which minimizes ECP(N) is bounded as follows:

$$N^* \leq N \leq N^{**} ,$$

where N^* minimizes the lower bound and N^{**} minimizes the upper bound given in (56).

The relationships for determining the optimal values of N for the lower and upper bounds are derived below.

Theorem IV-4. The optimal value of N which minimizes the lower bound is the largest value of N which satisfies

$$(N-1)(N-2)p^2 + 2[(N-2)p + 1] < \frac{2C_T p}{C_D} . \tag{65}$$

The optimal value of N which minimizes the upper bound is the largest value of N which satisfies

$$N(N-1)p + 2(1-p) < \frac{2C_T}{C_D} . \tag{66}$$

Proof. For the lower bound we want the largest N such that

$$ETC(N) \left[\frac{p}{1+(N-1)p}\right] - ETC(N-1) \left[\frac{p}{1+(N-2)p}\right] < 0 . \tag{67}$$

The difference in the C_T term is

$$C_T \left[\frac{p}{1+(N-1)p} - \frac{p}{1+(N-2)p}\right]$$

$$= \frac{-C_T p^2}{[1+(N-1)p][1+(N-2)p]} . \tag{68}$$

The difference in the C_D term is

$$C_D \left[\frac{p(N-1)\left[\frac{N-2}{2} p+1\right]}{1+(N-1)p} - \frac{p(N-2)\left[\frac{N-3}{2} p+1\right]}{1+(N-2)p}\right]$$

$$= C_D p \left[\frac{(N-1)(N-2)p^2 + 2(N-2)p + 2}{[1+(N-1)p][1+N-2)p]}\right] \tag{69}$$

The sum of (68) and (69) constitutes the left side of (67). Upon cancellation of common terms and movement of the cost parameters to the right side, we get the result shown as (65).

For the upper bound we assume that

$$\min \{\frac{1}{N} , - \ell np\} = \frac{1}{N}$$

for the N values where the conjecture would be useful. We then want the largest value of N for which

$$ETC(N)\left[\frac{p}{N(1-p)}\right] - ETC(N-1)\left[\frac{p}{(N-1)(1-p)}\right] < 0 \; . \tag{70}$$

The difference in the C_T term is

$$\frac{C_T p}{(1-p)}\left[\frac{1}{N} - \frac{1}{N-1}\right] = \frac{-C_T p}{N(N-1)(1-p)} \tag{71}$$

The difference in the C_D term is

$$\frac{C_D p}{2(1-p)}\left[\frac{(N-1)}{N}[(N-2)p + 2] - \frac{(N-2)}{(N-1)}[(N-3)p + 2]\right]$$

$$= \frac{C_D p}{2N(N-1)(1-p)}[pN(N-1) + 2(1-p)] \; . \tag{72}$$

The sum of (71) and (72) constitutes the left side of (70). Cancellations and rearrangements yield (66).

Modification of the Expected Costs per Period--The inclusion of the penalty cost for cancelling a scheduled delivery is not appropriate for this alternative since it is not a scheduled delivery strategy. As in Chapter III, it might be argued that the C_T value would be higher for this alternative than for Alternative 1. It might also be argued that the C_T value is lower than for Alternative 2 since the time at which a truck will be needed is known as soon as the first demand occurs after the truck completes the previous delivery. Thus a reservation can be made ahead of time.

The second modification was the inclusion of the fixed cost S per unit demanded. As with Alternative 1, the total expected costs ETC(N) over N periods is the product of S and the expected number delivered. And for Alternative 3, the expected number delivered, M_3, is:

$$M_3 = \sum_{x=1}^{N} x \binom{N-1}{x-1} p^{x-1} (1-p)^{N-x}$$

$$= \sum_{u=0}^{N-1} [u+1] \binom{N-1}{u} p^{u} (1-p)^{N-1-u}$$

$$= 1 + E(u)$$

$$= 1 + (N-1)p \tag{73}$$

The modified form of ETC(N) is given by equation (74).

$$ETC(N) = C_T + S + (N-1)\left[\frac{C_D(N-2)p}{2} + Sp + C_D\right] . \tag{74}$$

The lower bound for ECP(N) when the S term is included is:

$$\frac{pC_T + pC_D(N-1)\left[\frac{(N-2)p}{2} + 1\right]}{1 + (N-1)p} + Sp \tag{75}$$

and, if the value of C_D is negligible with respect to S, the optimal value of N for this bound is infinite. A similar argument results in an infinite optimal N for the upper bound. As a consequence, the N minimizing ECP(N) must also

be infinite. The reader will recall that this was also the result for Alternative 1.

Time Standards--With T being the time constraint in periods we again must have the expected delay per unit such that

$$UD(N) \leq T ,$$

where UD(N) is the expected delay per unit.

Theorem IV-5. The expected delay per unit under Alternative 3 is

$$UD(N) = \frac{1}{2}\left[(N-2) + \frac{[1-(1-p)^N]}{p}\right] \tag{76}$$

Proof. Following the arguments in the proof of Theorem II-3,

$$UD(N) = \sum_{x=1}^{N} \frac{TD(x,N)}{x} P(x,N)$$

$$= \sum_{x=1}^{N} \left[\frac{N}{2x} + \frac{(N-2)}{2}\right]\binom{N-1}{x-1} p^{x-1}(1-p)^{N-x}$$

$$= \frac{N}{2} E\left(\frac{1}{x}\right) + \frac{N-2}{2} \tag{77}$$

Now

$$NE\left(\frac{1}{x}\right) = N \sum_{x=1}^{N} \frac{1}{x}\binom{N-1}{x-1} p^{x-1}(1-p)^{N-x}$$

$$= \frac{1}{p} \sum_{x=1}^{N} \binom{N}{x} p^{x}(1-p)^{N-x}$$

$$= \frac{1-(1-p)^{N}}{p} \tag{78}$$

Substitution of (78) into (77) results immediately in (76).

In contrast to Alternative 1, substitution of (76) for UD(N) in the time constraint does not provide a simple expression for N.

Truck Capacity Constraint--If M is the maximum capacity of the delivery truck then the truck capacity constraint is $N \leq M$ since it is possible that N units will be demanded over the N periods.

V. ON-SITE SYSTEM

Introduction--An on-site inventory system is an alternative to delivery. In comparing it to direct delivery, it is appropriate to develop the comparable total expected costs per periods. Therefore, an expected costs model for the on-site system must be developed for comparison with each of the three direct delivery alternatives.

A basic assumption of these models will be that sufficient inventory exists on-site to meet all demands.

Cost Elements--A fixed cost per period is assumed to represent labor and related overhead costs. It will be denoted by C_L .

A delay cost is also appropriate since it is assumed that the on-site system will not be distributed to each repair production line but will instead be centrally located at the NARF. This delay cost is assumed to be a fixed cost per unit demanded and will be denoted by s .

Alternative 1 Model--In comparing the expected costs of the on-site system with the scheduled delivery alternative, we need to evaluate the expected delay costs over the same time frames as were used in Alternative 1. If we have k-1 intervals of N periods before a demand occurs then the probability of such an occurrence is given by formula (18). The expected number of units demanded during the last interval of N periods is given by equation (23) and the expected total delay costs are

$$sM_1 = \frac{sNp}{[1 - (1-p)^N]} \tag{79}$$

The expected delay costs per period over the kN periods is then

$$\frac{sM_1}{kN} = \frac{sp}{k[1 - (1-p)^N]} \ . \tag{80}$$

The expected value of (80) over all possible k values is determined from

$$\frac{sp}{[1 - (1-p)^N]} \sum_{k=1}^{\infty} \frac{1}{k}(1-p)^{(k-1)N}[1 - (1-p)^N]$$

$$= sp \sum_{k=1}^{\infty} \frac{1}{k}(1-p)^{(k-1)N}$$

$$= sp \left[\frac{-\ln[1 - (1-p)^N]}{(1-p)^N}\right] \tag{81}$$

Finally, the total expected costs per period for the on-site system, OECP(N), is the sum of C_L and (81); namely,

$$OECP(N) = C_L + sp \left[\frac{-\ln[1 - (1-p)^N]}{(1-p)^N}\right] . \tag{82}$$

Alternative 2 Model--The C_L term is again appropriate. The total delay cost is now the product sK for one delivery cycle. The delay costs per period, where n is the number of periods required to accumulate K demands, is

$$\frac{sK}{n} . \tag{83}$$

The probability of n periods being required is given by (28).

Combining (28) and (83) and summing over all possible n values gives

$$sK \sum_{n=K}^{\infty} \frac{1}{n}\binom{n-1}{K-1} p^K (1-p)^{n-k} . \tag{84}$$

And the total expected costs per period is the sum of C_L and (84); that is,

$$OECP(K) = C_L + sK \sum_{n=K}^{\infty} \frac{1}{n}\binom{n-1}{K-1} p^K (1-p)^{n-k} . \tag{85}$$

<u>Alternative 3 Model</u>--The expected number of units demanded before delivery under Alternative 3 is given by (73). Therefore, the total delay costs are

$$s[1 + (N-1)p]$$

The number of periods between deliveries is $(k-1) + N$ so that the delay cost per period is

$$\frac{s[1 + (N-1)p]}{(k-1) + N} \tag{86}$$

and the probability of the first demand occurring in period k is $p(1-p)^{k-1}$. The expected value of (86) over all possible k values is therefore

$$s[1 + (N-1)p] \sum_{k=1}^{\infty} \frac{p(1-p)^{k-1}}{N + k - 1}$$

Finally, the expected total costs per period is

$$OECP(N) = C_L + s[1 + (N-1)p] \sum_{k=1}^{\infty} \frac{p(1-p)^{k-1}}{N + k - 1} \tag{87}$$

VI. TRADE-OFF MODELS

Introduction--The first trade-off models to be presented below compare each direct delivery alternative with its associated on-site model. These models include the special conditions where optimal N and K are unity since they appear to correspond to situations under which an on-site system seems most appropriate (i.e., having an on-site system seems better intuitively than making a direct delivery every time a demand occurs). Finally, the models incorporate the time standard and truck capacity constraints.

The individual models are then combined into a composite model which is designed to resolve the question of which direct delivery alternative to use if direct delivery is the optimal strategy.

Alternative 1--When comparing the direct delivery costs given by (15) with the corresponding on-site costs given by (82), it follows that the on-site system is preferred when OECP(N) < ECP(N). When the cost elements are introduced into that inequality, the result is that

$$C_L < \left[\frac{C_T[1-(1-p)^N]}{N} + \frac{[C_D(N-1) - 2s]p}{2}\right]\left[\frac{-\ell n[1-(1-p)^N]}{(1-p)^N}\right] . \qquad (88)$$

Now (15) is minimized by a certain N value. However, the right side of (88) is minimized for a slightly larger value of N because of the (-2s) term creating a savings in the delay costs. It is this latter value that is appropriate

when considering the trade-off between on-site and direct delivery.

To have $N = 1$ minimize the right side of (88) we need to have the latter's value be less for $N = 1$ than for $N = 2$. When $N = 1$, (88) reduces to

$$C_L < \left[C_T - s\right]\left[\frac{-p\ell np}{1-p}\right] . \tag{89}$$

When $N = 2$, the right side of (88) is reduced as follows:

$$\left[\frac{C_T}{2}\,[1-(1-p)^2] + \frac{C_D p}{2} - sp\right]\left[\frac{-\ell n[1-(1-p)^2]}{(1-p)^2}\right]$$

$$= \left[\frac{C_T p}{2}\,(2-p) + \frac{C_D p}{2} - sp\right]\left[\frac{-\ell np(2-p)}{(1-p)^2}\right]$$

$$= \left[C_T p - sp - \frac{C_T p^2}{2} + \frac{C_D p}{2}\right]\left[\frac{-\ell np(2-p)}{(1-p)^2}\right]$$

$$= \left[(C_T - s) + \frac{1}{2}\left(C_D - C_T p\right)\right]\left[\frac{-p\ell np(2-p)}{(1-p)^2}\right] \tag{90}$$

Now (90) is larger than the right side of (89) if

$$C_D > C_T p + 2(C_T - s)\left[\frac{(1-p)\,\ell np}{\ell np(2-p)} - 1\right] . \tag{91}$$

Therefore, if C_D is large enough that (91) is satisfied and C_L is small enough that (89) is satisfied then an on-site system is optimal. If C_D does not satisfy (91), we must

compute the right side of (88) for several values of N to see which value minimizes it and then compare C_L with the resulting minimum. If C_L does not satisfy (88) or (89) then the on-site system is not preferred over the direct delivery strategy identified as Alternative 1 and the latter should then be considered for use.

The initial step in this consideration is to determine the optimal N value which minimizes equation (15). Next we must consider the effect of the constraints of time standards and truck capacity.

In the derivations of the direct delivery models the time standard T was assumed to be measured in periods. In reality, it is measured in hours or days. Therefore, given a time standard and the production schedule we can convert the actual time standards into equivalent periods.

We can combine the time constraint given by (27) and truck capacity constraint as follows:

$$N \leq \min\{M, 2T + 1\} , \tag{92}$$

where M is the truck capacity in units of the repair part in question.

If the on-site system was not preferred based on (88) or (89), then the optimal N determined from minimizing (15) must be tested against (92). If it does not satisfy (92), we select the largest integer value of N which does. We must then test to see if C_L satisfies (88) when this N value is introduced into the right side of the inequality. If C_L

now satisfies (88), the on-site system is preferred over the constrained direct delivery strategy. Otherwise, we use Alternative 1 under the constrained value of N.

Alternative 2--The approach in developing the trade-off model for Alternative 2 is similar to that described above for Alternative 1.

The general conditions for which OECP(K) < ECP(N), where equations (85) and (35) respectively apply, is given by (93).

$$C_L < (C_T - sK) \sum_{n=K}^{\infty} \frac{1}{n}\binom{n-1}{K-1} p^K (1-p)^{n-K} + \frac{C_D(K-1)}{2} . \tag{93}$$

When K = 1, (93) reduces to the inequality given by (94).

$$C_L < (C_T - s)\left[\frac{-p\ell np}{1-p}\right] . \tag{94}$$

The right side of (93) when K = 2 is

$$(C_T - 2s) \sum_{n=2}^{\infty} \frac{1}{n}\binom{n-1}{1} p^2 (1-p)^{n-2} + \frac{C_D}{2}$$

$$= (C_T - 2s) \sum_{n=2}^{\infty} \frac{n-1}{n} p^2 (1-p)^{n-2} + \frac{C_D}{2}$$

$$= (C_T - 2s) \frac{p}{1-p} \sum_{n=2}^{\infty} \left(1 - \frac{1}{n}\right) p(1-p)^{n-1} + \frac{C_D}{2}$$

(Continued)

$$= (C_T - 2s)\ \frac{p}{1-p}\left[1 - p - \frac{p}{1-p}\left(-\ell np - (1-p)\right)\right] + \frac{C_D}{2}$$

$$= (C_T - 2s)\ \frac{p}{1-p}\left[1 + \frac{p\ell np}{1-p}\right] + \frac{C_D}{2}\ . \tag{95}$$

The conditions under which (95) is greater than the right side of (94) are expressed by (96) and correspond to $K = 1$ minimizing the right side of (93).

$$C_D > \frac{2p}{(1-p)^2}\left[C_T[-\ell np - (1-p)] - s[(1+p)(-\ell np) - 2(1-p)]\right]\ . \tag{96}$$

If C_D satisfies (96) and C_L satisfies (94) then the on-site system is optimal. If C_D does not satisfy (96) then we must find that K which minimizes the right side of (93), evaluate the corresponding value of the right side, and then test for the value of C_L. If C_L now satisfies (93) then an on-site system is optimal. Otherwise, direct delivery under Alternative 2 should be examined by first determining the optimal value of K being that which minimizes (35) and then checking this K against the constraints.

The time standard and truck capacity constraints under Alternative 2 can be combined as

$$K \leq \min\{M, 2pT + 1\} \tag{97}$$

where M is the truck capacity in units of the repair part in question and T is the time standard in periods. The feasible value of K is the largest integer value satisfying

(97). If it is at least as large as that K minimizing (35) then the problem is solved. If it is not as large then we must proceed, as described for Alternative 1, with introducing the constrained K value first into (93) and testing for C_L. If C_L now satisfies (93) then the on-site system is preferred. Otherwise, direct delivery under Alternative 2 with the constrained value of K is optimal.

Alternative 3--The general conditions for which OECP(N) < ECP(N) when equations (87) and (55) are introduced are given by (98),

$$C_L < \left[C_T + C_D(N-1)\left[\frac{N-2}{2}\,p + 1\right] - s[1 + (N-1)p]\right]\left[\sum_{k=1}^{\infty} \frac{p(1-p)^{k-1}}{N + k - 1}\right]. \tag{98}$$

When $N = 1$, (98) reduces to

$$C_L < (C_T - s)\left[\frac{-p\ell np}{1-p}\right]. \tag{99}$$

When $N = 2$, the right side of (98) is

$$\left[C_T + C_D - s(1+p)\right] \sum_{k=1}^{\infty} \frac{p(1-p)^{k-1}}{k+1}$$

$$= [C_T + C_D - s(1+p)]\,\frac{p}{(1-p)^2}\,[-\ell np - (1-p)] \tag{100}$$

The conditions under which $N = 1$ minimizes the right side of (98) are obtained by comparing (100) with the right side of (99). The result is:

$$C_D > C_T\left[\frac{(1-p) + p\ell np}{-\ell np - (1-p)}\right] + s\left[\frac{p^2 - 2\ell np - 1}{-\ell np - (1-p)}\right] \tag{101}$$

Therefore, if C_D satisfies (101) and C_L satisfies (99) then an on-site system is optimal. Otherwise, direct delivery under Alternative 3 is considered further.

The time standard and capacity constraints for Alternative 3 cannot be combined because equation (76) for the expected delay per unit is not a simple function of N. As a consequence, the unconstrained optimal value of N for Alternative 3 must be tested against

$$N \leq M ,$$

and

$$\frac{1}{2}(N-2) + \frac{[1 - (1-p)^N]}{p} \leq T .$$

If the constraints are satisfied by unconstrained optimal N then the solution is direct delivery under Alternative 3 with this value of N. If the constraints are not satisfied then we must select the largest integer value for N which satisfies the constraints and re-examine (93). If C_L satisfies (93) an on-site system is preferred. Otherwise we use direct delivery with constrained N.

The Composite Model--The goal of the trade-off analyses is the determining of the best strategy when all three direct delivery alternatives are considered together (the composite problem). This is done by first obtaining the best solution

for each alternative's trade-off model and then comparing these solutions.

Obviously, if an on-site system is optimum under all three alternatives, then an on-site system is the solution to the composite problem. If, however, direct delivery is best in all three cases then the optimal solution to the composite problem is that direct delivery alternative which provides the lowest optimal total expected costs for the given C_T , C_D , and p values.

Now suppose that one alternative's trade-off model gives an on-site system as optimal but another does not. When the two results are compared, that alternative favoring direct delivery is automatically preferred since it was preferred over an on-site system when it was considered by itself. As a consequence, if only one direct delivery alternative was found optimum under the individual trade-off models, it is the optimal solution to the composite problem. If two trade-off models favor direct delivery and the third does not, then it follows that the optimal solution to the composite problem is obtained from comparing the minimum expected total costs per period for the two models favoring direct delivery.

VII. SUMMARY AND RECOMMENDATIONS

Summary--Trade-off models for deciding on where to place inventories for a given repair part needed by a NARF for a certain aircraft or component rework have been developed in the preceding chapters. Two locations for the inventory were considered; at the NSC and at the NARF. Splitting of the inventory between the two locations was not considered in keeping with the current philosophy of the NSCs towards NARF support. When the location was assumed to be at an NSC, three direct delivery alternatives were considered and included both scheduled and unscheduled delivery.

Expressions for the total expected costs per period were derived for all alternatives to provide a basis for comparison. The total cost was assumed to consist of a delivery cost and a production delay cost for direct delivery. For the on-site inventory system at the NARF, the costs were assumed to consist of a labor charge and a delay cost. Constraints were also developed to reflect the impact of time standards and delivery truck capacity.

Unfortunately, the complexity of the various expected cost expressions did not allow for optimal solutions to be derived analytically. Therefore, algorithms were developed for using the models to resolve the question of where to site the inventory.

Recommendations--Further understanding of the models should be obtained through parametric studies. Some of these are

well underway and suggest that between the direct delivery alternatives the optimal expected costs per period differ little in value. Additional studies are however needed and are being planned.

As more of an understanding of the models is obtained, refinements will undoubtedly appear appropriate. In fact, the chapters addressing the direct delivery models contain some preliminary refinements in the expected delay costs expressions which were motivated by the parametric studies which have already been done. Additional issues relative to delivery costs for scheduled versus unscheduled delivery have also been raised in Chapters II and IV. The question of what is an appropriate delay cost to assume for an on-site system also needs further study.

As was mentioned earlier, the models have been restricted to a given repair part for a certain production line at the NARF. Expansion of these models needs to be done to include multiple sources of demand within the NARF for the parts.

Finally, an interesting additional expansion which also seems appropriate is the case where two or more repair parts are forced to have the same decision variable value in direct delivery; in particular, the same time between scheduled deliveries. This would correspond to a more realistic scheduled delivery scheme than one designed for each part.

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